DOCUMENT RESUME

ED 167 382 SE 026 757

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TITLE Three-Year Sequence for High School Mathematics, Course III.
INSTITUTION New York State Education Dept., Albany. Bureau of General Education Curriculum Develcpment.; State Univ. of New York, Albany.
PUB DATE Jan 79
NOTE 44p.; For related document, see ED 134 473

EDRS PRICE MF-$0.83 HC-$2.06 Plus Postage.
DESCRIPTORS Curriculum; *Grade 11; *Instruction; Secondary Education; *Secondary School Mathematics; *Teaching Guides; *Units of Study
IDENTIFIERS *New York

ABSTRACT

This curriculum guide covers Course III of a three-year sequence for high school mathematics in New York which was intended to provide an alternative to the regular Regents sequence of ninth-, tenth-, and eleventh-grade mathematics. A listing of scope and content is provided along with suggested time allotments. Topics covered are: complex numbers, relations and functions, circular functions, transformation geometry, and probability and statistics. Material in each chapter is discussed and some teaching suggestions are made. (MP)

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THREE-YEAR SEQUENCE FOR HIGH SCHOOL

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mathematics

COURSE III

The University of the State of New York
THE STATE EDUCATION DEPARTMENT
Bureau of General Education Curriculum Development
Albany, New York

SE 026 757

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FOREWORD

This is the third in a series of three publications designed for the new *Three-Year Sequence for High School Mathematics* program consisting of three unified courses - *Course I*, *Course II*, and *Course III*. Currently, this sequence provides an alternate to the regular Regents sequence of ninth, tenth, and eleventh year mathematics. Examinations will be available each year for each course, and students who successfully complete the courses will be credited with a Regents sequence in mathematics.

In June 1972, an Advisory Committee was convened by Frank Hawthorne, then Chief of the Bureau of Mathematics Education, to consider the development of a three-year high school mathematics program which would bring together the various branches of mathematics that historically have been treated independently. Members of this committee included: Robert Atkins, John Adams High School, New York City; Donald Holquist, then of Mineola High School, now at Baldwin High School; Anthony Prindle, Linton High School, Schenectady; Ralph Roberts, Pelham Memorial High School; Harry Ruderman, Hunter College High School; George Salterelli, Amherst High School; Stephen Willoughby, New York University; and Sister Ann Xavier, Nazareth High School, Rochester.

Using the philosophical deliberations and tentative outline developed by the committee, *Course III* was organized and written by Sister Ann Xavier of the committee and Richard Klutch of Hunter College High School, working under the direction of Fredric Paul, Chief of the Bureau of Mathematics Education. John Matmillo of the Wappingers Central Schools reviewed the material. In September 1976, 40 school districts began offering the course.

As a result of a series of meetings with classroom teachers during 2 years of teaching the course, together with their written evaluations at the end of that time, *Course III* was revised during the summer of 1978. Janet Burt of Clinton High School and Sherman Taishoff of Harrison High School analyzed the evaluations and made recommendations which were incorporated into the revision of the materials by the original writers.

The final draft was prepared and edited by Mr. Paul and Aaron Buchman, Associate in Mathematics Education. Robert F. Zimmerman, Associate in Secondary Curriculum, represented the Bureau of General Education Curriculum Development on the committee and served as the liaison with the Bureau of Mathematics Education. The final manuscript was prepared in the Bureau of General Education Curriculum Development.

January 1979

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COURSE III

Scope and Content

I. Complex Numbers (recommended time: 25 days)

- A. Further study of the real number system
- B. Extension of algebraic techniques
- C. Introduction of the complex number system
- D. Quadratics with complex roots
- E. Absolute value: equations and inequalities

II. Relations and Functions (recommended time: 25 days)

- A. Definitions and properties
- B. Exponential and logarithmic functions
- C. Common logarithms and computation

III. Circular Functions (recommended time: 60 days)

- A. Further work with the geometry of the circle
- B. Circular functions and their graphs
- C. Trigonometric identities and equations
- D. Applications to area, Law of Sines, Law of Cosines, right triangle trigonometry

IV. Transformation Geometry (recommended time: 20 days)

- A. Review of mapping
  1. Definitions, one-to-one, identity mappings, inverse mappings
  2. Some geometric examples
- B. Line Reflections
  1. Considered as mappings
  2. Properties "preserved" by line reflections
  3. Line symmetry
  4. Coordinates of image of point (x, y) under reflection in special lines of coordinatized plane

C. Translations

1. Considered as mappings
2. Properties "preserved" by translations
3. Translations as composition of line reflections
4. Coordinates of image of point (x, y) under translation

D. Rotations

1. Considered as mappings
2. Properties "preserved" by rotations
3. Rotations as composition of line reflections
4. Coordinates of image of point (x, y) under special rotations
5. Rotational symmetry

E. Isometries

1. Definition
2. Relation to congruence
3. Direct isometries
4. The three-line reflection theorem
5. Dilations as examples of transformations which are not isometries
6. Coordinates of image of point (x, y) under dilation
7. Relation to symmetry

F. Applications to Group Theory

1. The identity transformation
2. Transformations whose composition is the identity
3. The group of translations
4. The group of rotations
5. The groups of isometries; of direct isometries

V. Probability and Statistics (recommended time: 20 days)

A. Review of previous work with probability, combinatorics

B. Probability of exactly r successes in n trails

C. The Binomial Theorem

D. Review of previous work with statistics

E. Measures of dispersion:

1. range, mean absolute deviation
2. variance
3. standard deviation

F. Σ - notation

VI. Optional Topics

A. Polar coordinates

B. Number theory

C. Mathematical induction

D. Sequences and series

E. Matrices

F. Synthetic substitution, remainder theorem, and factor theorem

I. Complex Numbers

A. In *Courses I* and *II*, students investigated mathematical systems involving sets of elements, operations, and properties of these operations. In *Course III*, teachers may begin a more structure-oriented discussion of the real number system by considering the set of whole numbers (the set of natural numbers and zero). These may be regarded as undefined elements or may be defined in terms of some more basic concept, such as sets. Together with the operations of addition and multiplication, the whole numbers form a system in which the closure, commutative, associative, distributive, and cancellation properties are valid. Zero is the additive and one the multiplicative identity.

In this number system, subtraction is defined so that $a - b = x$ if, and only if, $b + x = a$. But in the system of whole numbers, the equation $b + x = a$ does not always have a solution, and, therefore, subtraction is not always possible. This motivates the introduction of the set of integers.

For all a in the set of whole numbers, $-a$, the additive inverse of a, is the solution of the equation $a + x = 0$. The union of the set of whole numbers and the set of their additive inverses is called the set of integers.

In order that the integers, as a number system, retain the basic properties of the whole number system, it is necessary to define addition and multiplication is some way similar to the following: for all a, b, whole numbers,

$$(-a) + (-b) = -(a + b)$$

$$(-a) + (b) = \begin{cases} b - a \text{ if } b - a \text{ is a natural number} \\ -(a - b) \text{ if } a - b \text{ is a natural number} \end{cases}$$

$$(-a)(-b) = ab$$

$$(-a)(b) = -(ab)$$

The equation, $b + x = a$, always has a solution and that solution can be shown to be $a - b = a + (-b)$.

Division is defined in a manner similar to subtraction. If a and b are integers and $b \neq 0$, the quotient $\frac{a}{b}$ is equal to x if and only if $bx = a$. But this equation does not always have a solution in the set of integers. Again, it is necessary to introduce a new set of numbers, the set of all ratios of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. Equality of elements in this set and operations of addition and multiplication are defined so that the properties of the whole numbers and of the integers hold:

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

This new set forms a number system in itself, some of whose elements are integers. To demonstrate that this is the case, consider the equation $1x = a$.

$x = a$, since 1 is the identity element for multiplication

$\therefore x = \frac{a}{1}$ by the definition of division

The integer a is equal to the ratio $\frac{a}{1}$

Also, consider the equation $cx = ca$, $c \neq 0$

$x = a$ by the cancellation law

$x = \frac{ca}{c}$ by the definition of division

$\therefore a = \frac{ca}{c}$

Using this new set of numbers, called the rational numbers, the equation $bx = a$ always has a solution if $b \neq 0$. In particular, the equation $ax = 1$ has a solution for all $a \neq 0$ and that solution, $\frac{1}{a}$, is called the multiplicative inverse (or reciprocal) of a. It is now possible to verify that the set of rational numbers under the operations of addition and multiplication form a field. (See *Course II*, page 18.)

While the transition from natural numbers and whole numbers to integers and from integers to rational numbers is relatively simple, the introduction of the real number system is not. If the approach used before is employed, the equation $x^2 = a$ could be considered. This equation does not always have a solution in the set of rational numbers. The introduction of a set of numbers which will solve this equation for $a > 0$ is very different from the set of numbers which will solve the equation for $a < 0$ and in neither case do we have a basis for the set of real numbers. So while an equation such as $x^2 = 2$ indicates the need for a set of numbers which includes numbers which are not rational, it does not help us to define the reals. At this level of mathematical maturity, most students do not have the

mathematical background for a complete definition. The most common approach is an appeal to the number line. The real numbers are described in terms of a 1 to 1 correspondence with the points on a line. Such a description could motivate a geometric interpretation of addition, subtraction, multiplication, and division. In addition, this approach might suggest an informal discussion of "greater than" in terms of "to the right of, on the number line."

A more formal approach to the ordering of the real numbers uses positive as an undefined term and the statement that the product of two positive numbers is a positive number as a postulate. Then, for all real numbers a and b, $a > b$ if and only if there exists a positive real number x such that $a = b + x$. Using this definition, the following statements can be proved.

If a is positive, then $a > 0$ since $a = 0 + a$

If $a > 0$, then $0 > -a$ since $0 = -a + a$

If $a > b$, then $a + c > b + c$ since $a = b + x \longrightarrow a + c = b + c + x$

If $a > b$ and $c > 0$, then $ac > bc$ since $a = b + x \longrightarrow ac = bc + xc$

If $0 > a$ and $c > 0$, then $0 > ac$ since $0 > a \longrightarrow 0 = a + x$

and $c > 0 \longrightarrow c = 0 + c$

$\therefore 0 = ac + xc$

If $a > b$ and $0 > c$ then $bc > ac$ since $a = b + x$

$ac = bc + xc$

$0 > xc$

$-xc > 0$

$ac + (-xc) = bc$

$bc > ac$

The teacher may find it convenient to introduce into this discussion the definition of negative (a is negative if and only if $0 > a$) and the definition of less than ($a < b$ if and only if $b > a$).

Another common characterization of the reals is in terms of "infinite" decimals. The student has had no previous work with sequences or series, and many important concepts involved in working with numbers in this form would have to be accepted without proof if the teacher chooses to present the real numbers

in this way. However, such a presentation can provide a relatively accessible distinction between rational and irrational numbers and highlight the existence of irrational numbers which are not radicals or in any way dependent on the solution of an algebraic equation.

B. As the discussion of the real number system develops, the teacher will find ample opportunity to use the properties under discussion to strengthen students' ability to rewrite algebraic expressions in simpler form. The distributive property of multiplication over addition should be familiar to the student and its use in factoring recognized. Students have had limited experience with algebraic fractions. Practice should lead to competence in writing equivalent fractions, multiplying and dividing algebraic expressions in fraction form, and in the simplification of complex fractions. Students should be able to simplify fractions involving radicals and, in particular, should be able to rationalize the denominator of a fraction.

C. The complex number system can be introduced in a manner similar to that used for the integers and the rational numbers. The fact that $x^2 = -1$ has no solution in the set of real numbers, motivates the definition $i = \sqrt{-1}$. The set of complex numbers is defined as the set of all numbers of the form $a + bi$, where a and b are real numbers. The operations of addition and multiplication are defined so that the basic properties hold. Students should develop facility in working with these operations while verifying that the complex numbers, together with addition and multiplication as defined, form a field.

It should be brought to the student's attention that unlike the fields of rational numbers and real numbers the field of complex numbers cannot be ordered. Those complex numbers which are not real numbers cannot be classified as positive or negative even when the real numbers involved in the formation of the number can be so classified. For example, if 3i were positive or negative, its square would be positive, which, of course, it is not. Also, it is false that 5i is greater than 2i, since $5i - 2i = 3i$ is not positive.

D. The solution of quadratic equations by the quadratic formula was studied in *Course II*. At that time only equations with real solutions were considered. These equations should be reviewed and equations with complex roots should be considered. This study should include the use of the discriminant to determine the nature of the roots, the relationship between the quadratic equation and the second degree polynomial function and the relationship between the nature of the roots and the graph of the function.

It can be shown, either by examples of factored quadratics and/or computation with

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

that for any quadratic $ax^2 + bx + c = 0$, the sum of the roots equals $\frac{-b}{a}$ and the products of the roots equals $\frac{c}{a}$. Example: Write a quadratic equation having the roots $(3 - 2i)$ and $(3 + 2i)$.

$$x_1 + x_2 = \frac{-b}{a} \qquad x_1 \cdot x_2 = \frac{c}{a}$$

$$(3 - 2i) + (3 + 2i) \qquad (3 - 2i)(3 + 2i)$$

$$6 = \frac{-b}{a} \qquad 9 - 4i^2$$

Let $a = 1$ $\qquad 9 + 4$

then $b = -6$ $\qquad 13 = \frac{c}{a}$

since $a = 1$

$c = 13$

∴ An equation would be $x^2 - 6x + 13 = 0$

E. For students who have not already studied absolute values, it should be introduced and formally defined at this time.

Here is one definition:

$$\begin{cases} |x| = x & \text{if } x \geq 0 \\ |x| = -x & \text{if } x < 0 \end{cases}$$

This definition can be used to solve first degree absolute value equations.

Example: Solve: $|2x - 3| = 5$

If $2x - 3 \geq 0$	If $2x - 3 < 0$
$2x - 3 = 5$	$-(2x - 3) = 5$
$2x = 8$	$-2x + 3 = 5$
$x = 4$	$-2x = 2$
	$x = -1$

The solution set for the equation is $\{ -1, 4 \}$

Simple inequalities using absolute value can also be pursued at this time. Many teachers will wish to limit their discussions to problems like $| x | < 4$ or $| x | > 4$, the solution to which, with appropriate illustrations on the number line, can be related to problems involving quadratics (such as $x^2 < 16$). In other classes, teachers may extend this work to include more complex inequalities such as $| 2x - 3 | < 5$ or quadratic inequalities of the type $x^2 - 4x - 5 < 0$.

II. Relations and Functions

A. Throughout the study of mathematics, as well as in non-mathematical situations, students have encountered the concept of relations. In the section dealing with Euclidean geometry in *Course II*, this concept was introduced in order to discuss the idea of an equivalence relation. Since the concept of relation underlies and simplifies the expression of many mathematical ideas, it should now be considered in greater detail.

Each time use is made of a mathematical expression like $2x + 1$, a relation has been implied: associated with each element, x, there is a second element, $2x + 1$. Thus, a relation is often defined as a set of ordered pairs. The domain of the relation is the set of all first components of the ordered pairs. The range of the relation is the set of all second components.

A relation may be specified by actually listing the ordered pairs if the set is finite.

Example 1:

student	mark
Mary	87
Joe	82
Peter	95
Randy	76
Alice	68
Mark	82
Linda	92

The domain is {Mary, Joe, Peter, Randy, Alice, Mark, Linda} and the range is {87, 82, 95, 76, 68, 92}.

A relation is frequently specified by a rule which makes it possible to determine the ordered pairs. Students should be familiar with a variety of ways of expressing such a rule. Some possibilities follow:

$$F = \left\{ (x, y) \mid y = \frac{1}{\sqrt{x-1}} \right\}$$

$$F: x \to \frac{1}{\sqrt{x-1}}$$

$$F = \left\{ \left(x, \frac{1}{\sqrt{x-1}} \right) \right\}$$

$$x \xrightarrow{F} \frac{1}{\sqrt{x-1}}$$

$$F(x) = \frac{1}{\sqrt{x-1}}$$

If not specified, the universe is generally understood to be the set of real numbers. The domain of the relation F, shown above, is the set of all reals greater than 1, i.e., $\{x \mid x > 1\}$, and the range is the set of all positive reals, i.e., $\{y \mid y > 0\}$.

A function is a special kind of relation in which each element of the domain corresponds to exactly one element of the range. In other words, a function is a relation in which no two ordered pairs have the same first element. F, shown above, is a function.

A function is a one-to-one function if and only if each second element corresponds to a unique first element. The set of ordered pairs obtained by interchanging the first and second elements of each pair of a one-to-one function, F, is the inverse function, F^{-1}.

The idea of inverse is familiar to students in different settings. They have seen that using the inverse "will undo what has been done:"

-3 is the additive inverse of 3 and $4 + 3 = 7$ and $7 + (-3) = 4$

2 is the multiplicative inverse of $\frac{1}{2}$ and $6 \times \frac{1}{2} = 3$ and $3 \times 2 = 6$

It seems reasonable, then, to consider f^{-1} as the inverse of f if, for all x, $x \xrightarrow{f} y$ and $y \xrightarrow{f^{-1}} x$

Example: If $f(x) = 2x + 1$, then $f^{-1}(x) = \frac{x-1}{2}$

$$x \xrightarrow{f} 2x + 1 \text{ and } 2x + 1 \xrightarrow{f^{-1}} x$$

The result of interchanging the first and second elements of each pair of a function which is not one-to-one is not a function and, therefore, is not an inverse. Consider, for example,

the function $F(x) = y = x^2$. If x and y are interchanged, $x = y^2$ or $y = \pm\sqrt{x}$. Call this relation G. Then,

$x \xrightarrow{F} x^2$ and $x^2 \xrightarrow{G} \pm x$. Starting with x, the result of applying F and then G can be either x or $-x$. Therefore G did not necessarily "undo what had been done" by F and is not the inverse of F.

A common method of producing an inverse function when the original function is not one-to-one is to restrict the domain of the given function to make it one-to-one.

Example 1: $F: x \longrightarrow x^2$ Let the domain of F be the set of non-negative real numbers.

$F^{-1}: x \longrightarrow \sqrt{x}$ F^{-1} is the inverse of F. The domain of F^{-1} is the range of F, the set of non-negative real numbers.

When logarithms are introduced, there is the following example.

Example 2: $F: x \longrightarrow 2^x$

$F^{-1}: x \longrightarrow \log_2 x$

Since F is one-to-one, F^{-1} is the inverse function. Its domain is the range of F, the positive reals.

After the introduction of the trigonometry functions, there is the following example.

Example 3: $G: x \longrightarrow \sin x$

$H: x \longrightarrow \arcsin x$

Since G is not one-to-one, H cannot be an inverse of G, unless the domain of G is restricted.

$G: x \longrightarrow \sin x \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$G^{-1}: x \longrightarrow \text{Arc sin } x$

The range of G^{-1} is $\{-\frac{\pi}{2} \leq \text{Arc sin } x \leq \frac{\pi}{2}\}$

B. Two important types of functions are the exponential functions and their inverses, the logarithmic functions. Students have already encountered algebraic expressions involving exponential notation and have studied the rules for operating with them. These can be briefly reconsidered before introducing the usual definitions for a^n (n rational).

$a^1 = a$; $a^n = \underbrace{a \cdot a \cdot a \cdot a \cdots a}_{n \text{ factors}}$; $a^0 = 1$, $a \neq 0$

$a^{-n} = 1/a^n$

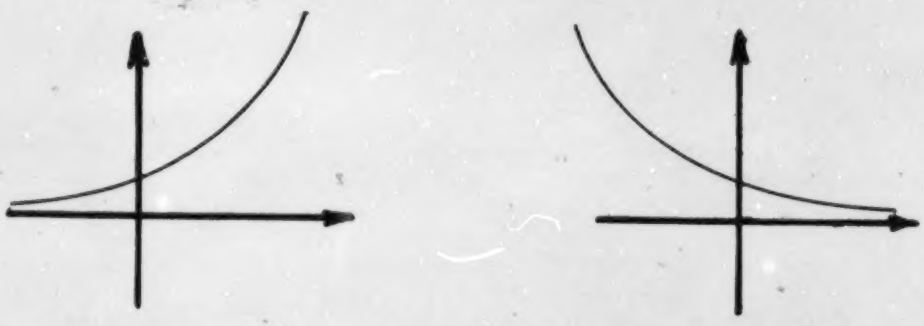
$a^{1/n} = \sqrt[n]{a}$

$a^{m/n} = (a^{1/n})^m$, $a > 0$

} m, n natural numbers

Since complex numbers were introduced in section 1, $a^{1/n}$ names a number with which students are familiar even in the case $a < 0$. However, $(a^{1/n})^m = (a^m)^{1/n}$ requires that $a > 0$. (For example, $[(-4)^{\frac{1}{2}}]^3 \neq [(-4)^3]^{\frac{1}{2}}$). Teachers may wish to justify these definitions as they are presented by appeal to the basic laws of exponents. Teachers should include in their development a discussion of scientific notation and the use of exponents to express very large or very small numbers.

An exponential function may be defined as $\{(x, y) \mid y = b^x\}$ where b is a positive real number different from 1. It is suggested that students be asked to sketch the graphs of several exponential functions since their nature is often better understood in this way. Below are sketches of $y = 2^x$ and $y = (\frac{1}{2})^x$. Of course, the graph of $y = a^x$, for any nonzero a will contain the point (0, 1).



The calculation of various values in the interval from -1 to 1 might be emphasized. It is suggested that teachers discuss the meaning of the expression "exponential growth" and its application to investigations in the physical and social sciences.

If $y = a^x$ then $x = \log_a y$. If $f(x) = a^x$, then $g(x) = \log_a x$ is the inverse of f. Students should graph some simple logarithmic functions such as $y = \log_2 x$ and compare them with the graphs of their inverses. It is hoped that the emphasis on the functional and graphic aspects of the exponential and logarithmic functions will be helpful to students in their future study of mathematics.

C. Logarithms to the base ten, or common logarithms, have historical importance as computational aids and as the basis of the slide rule. While electronic calculating devices are replacing such logarithmic computation, students should understand the principles of these computations and have some experience with them.

III. Circular Functions

A. In previous courses, the student has defined angle, congruent angles, and degree measure of an angle. It is useful to define degree measure of a minor arc of a circle as equal to that of the central angle whose rays contain the endpoints of the arc. The wording of this concept and those which follow from it should be consistent with that used by students in the study of angles in *Course II*. The definitions of chord, secant, and tangent should be introduced or reviewed and the measures of the angles formed when these line segments intersect should be related to the measures of their intercepted arcs.

Using these results of arc and angle measure, it is possible to prove theorems related to the circle which make use of similar triangle relationships. The proofs of theorems such as the following may be useful in themselves and in reviewing the concept of similarity.

- If two chords intersect within a circle, the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other.
- Two tangent segments drawn to a circle from an external point are congruent.
- If a tangent and a secant are drawn to a circle from an external point, the measure of the tangent segment is the mean proportional between the measure of the secant and the measure of its external segment.

Most of the previous work with angles has been concerned with a static concept of angles in convex polygonal figures and thus limited to angles whose measures are less than that of a straight angle. But it is also useful to consider a dynamic interpretation of angle in terms of rotation. Such a definition makes it possible for any real number to be the measure of an angle.

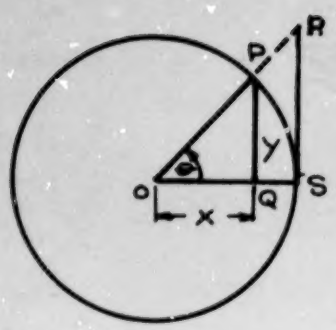
While degree measure is convenient in relating angle or arc size to that of a complete rotation, it is often more important to use a measure which relates angle or arc measure to linear measure. If a central angle of a circle intercepts an arc equal in length to the radius of the circle, then the measure of the angle is one radian. In particular, in a circle of radius one, a central angle of one radian intercepts an arc of one unit.

In this same circle, a central angle of n radians intercepts an arc of n units. Since the length of arc of one complete rotation is the circumference of the circle, $2\pi r$, or for a circle of radius one, 2π, a rotation of measure 360° is equivalent to that of 2π radians. Students should be able to express angle measure in terms of either degree or radian measure.

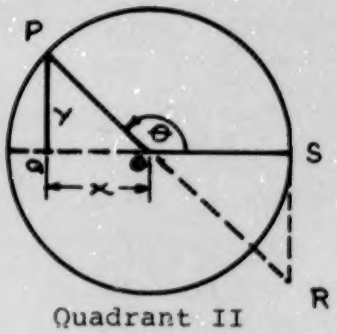
B. In the Cartesian coordinate system, a circle with center at the origin and radius one is designated by $\{(x, y) \mid x^2 + y^2 = 1\}$. If a ray coincident with the positive x-axis with endpoint at the origin is rotated through θ radians, a point on that ray one unit from the origin traces out a path on the unit circle of length θ. The final position of this ray, called the terminal side, determines both an angle, and an arc of the unit circle, of measure θ. Thus (1, 0) represents the zero position for measuring arc length along the unit circle. Motion in the counterclockwise direction is considered positive and motion in the clockwise direction considered negative. For each real number, θ, which represents a length of path on the unit circle, there is a point with coordinates (x, y) such that $x^2 + y^2 = 1$. The sine function is defined as the set of ordered pairs $\{(\theta, y)\}$ and the cosine function is defined as the set of ordered pairs $\{(\theta, x)\}$. That is, $\sin \theta = y$ means that $\sin \theta$ is the vertical distance from the x-axis to the endpoint of the arc of length θ on the unit circle and $\cos \theta = x$ means that $\cos \theta$ is the horizontal distance from the y-axis to the endpoint of that arc. The domain of each function is the set of real numbers and the range is the set of numbers from -1 to 1 inclusive.

Using these same definitions of x, y, and θ, $\tan \theta$ is defined to be $\frac{y}{x}$, $\sec \theta$ to be $\frac{1}{x}$, $\cot \theta$ to be $\frac{x}{y}$, and $\csc \theta$ to be $\frac{1}{y}$ for all x, y for which the functions are defined.

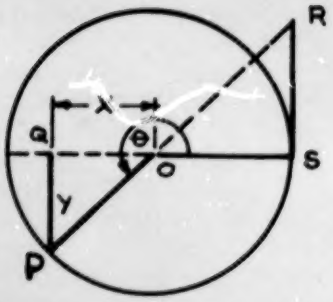
It is convenient to represent the function values for the sine, cosine, and tangent functions as the lengths of line segments related to the unit circle.



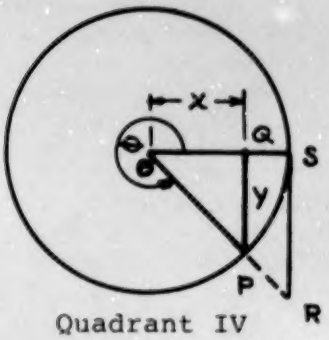
Quadrant I



Quadrant II



Quadrant III



Quadrant IV

In these diagrams, $\sin \theta = PQ$, $\cos \theta = OQ$. By definition, $\tan \theta = \frac{y}{x}$, and, since $\Delta OPQ \sim \Delta ORS$, $\frac{y}{x} = \frac{RS}{1}$ since these are the measures of corresponding parts of similar triangles. Therefore $\tan \theta = RS$. Students should be familiar with common angles of measure $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ radians, and their multiples, and will find it convenient to know or be able to determine the exact values of the sines, cosines, and tangents for these angles. Students should also be able to express the function value of any angle in terms of a function value of an acute angle.

The notation $\{(\theta, y)\}$ and $\{(\theta, x)\}$ used to define the sine and cosine functions is not the usual notation used in graphing and the student should understand that the change to the notation $y = \sin x$ and $y = \cos x$ in no way changes the concept involved. The significance of a and b in the expressions $y = a \sin bx$ and $y = a \cos bx$ should be considered and utilized in graphing these functions for simple positive values of a and b. The notion of <u>period</u> and <u>amplitude</u> should be introduced. The measure of the angle, x, should be expressed in radians and the scale on the x-axis made comparable to that on the y-axis. The graph of $y = \tan x$ should also be considered.

Consideration should be given to the inverse trigonometric functions, $y = \text{Arc sin } x$, $y = \text{Arc cos } x$, and $y = \text{Arc tan } x$.

C. An immediate consequence of the definitions of the tangent, cotangent, secant, and cosecant functions are the identities:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \csc \theta = \frac{1}{\sin \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \qquad \tan \theta = \frac{1}{\cot \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

Since the sine and cosine functions have been defined in terms of the unit circle, $x^2 + y^2 = 1$, it follows that:

$$\cos^2\theta + \sin^2\theta = 1$$

By dividing both sides of this equation by $\cos^2\theta$ or by $\sin^2\theta$, it follows that:

$$1 + \tan^2\theta = \sec^2\theta \text{ if } \cos \theta \neq 0$$

$$\cot^2\theta + 1 = \csc^2\theta \text{ if } \sin \theta \neq 0$$

These eight basic identities should be used to prove other trigonometric identities. Since many trigonometric identities involve possibly troublesome algebraic expressions such as complex fractions, the teacher may need to introduce or review the principles involved in working with such expressions as a part of this section. Attention should be given to equations involving trigonometric functions, including quadratics, some of whose solutions require use of the quadratic formula.

D. If P is any point on the unit circle, $x^2 + y^2 = 1$, such that the length of arc from the point (1, 0) to P is θ, then the coordinates of P are $(\cos \theta, \sin \theta)$. If P' is any point on the ray OP, then $\Delta OPQ \sim \Delta OP'Q'$. Therefore, $\frac{PQ}{OP} = \frac{P'Q'}{OP'}$

Since $PQ = \sin \theta$ and if $OP' = a$,

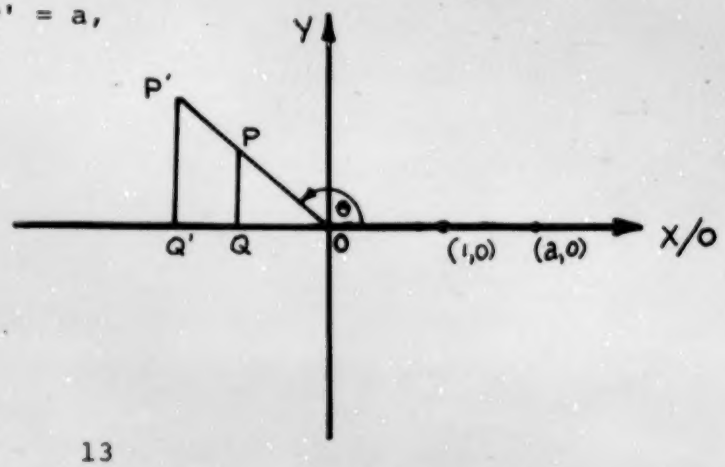
$$\frac{\sin \theta}{1} = \frac{P'Q'}{a}$$

$$P'Q' = a \sin \theta$$

and

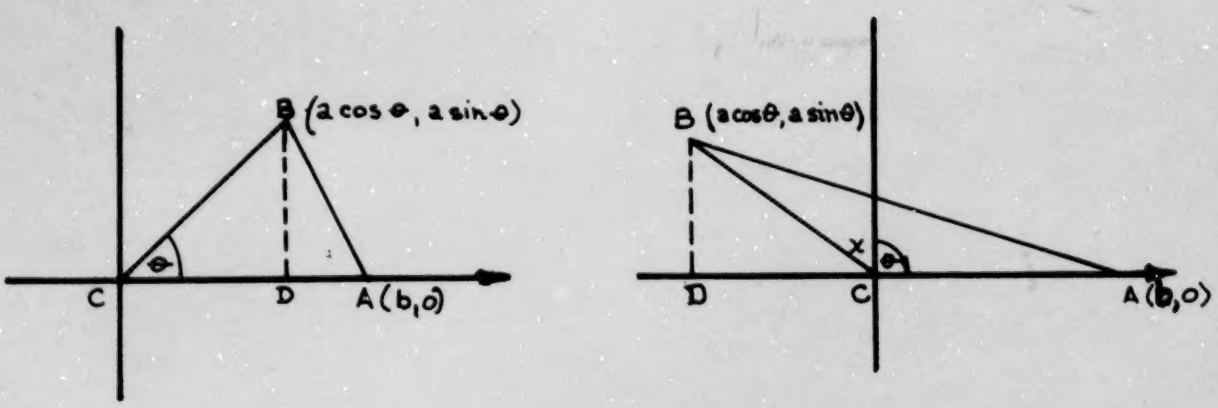
$$\frac{\cos \theta}{1} = \frac{OQ'}{a}$$

$$OQ' = a \cos \theta$$



Therefore the coordinates of P' are (a cos θ, a sin θ).

Any triangle may be associated with the coordinate system so that one vertex of the triangle is at the origin and one side of the triangle lies along the x-axis. Then the distance formula makes it possible to determine the length of one side in terms of the lengths of the other two sides and the measure of the angle between them.



$$c^2 = (b - a\cos\theta)^2 + (a\sin\theta)^2$$

$$c^2 = a^2\cos^2\theta - 2ab\cos\theta + b^2 + a^2\sin^2\theta$$

$$c^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2 - 2ab\cos\theta$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

or,

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{(Law of Cosines)}$$

Using these same triangles,

$$\text{Area of } \Delta ABC = \tfrac{1}{2}(CA)(BD) \text{ or,}$$

$$\text{Area of } \Delta ABC = \tfrac{1}{2}(b)(a\sin C)$$

By choosing the coordinate system appropriately, three expressions for area may be obtained:

$$\text{Area of } \Delta ABC = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}bc\sin A$$

If each member of this equality is divided by ½abc, the result is the Law of Sines:

$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

The student should have some experience in solving triangles and in working with physical and other applications of this material, such as those concerning resolution of forces.

The rules derived above apply to all triangles and therefore to right triangles. However, for problems involving right triangles – lighthouses, shadows, angles of elevation, etc., the teacher may wish to introduce the special relationship of right triangles,

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

which can be derived by applying the definitions of sine, cosine, and tangent to a first quadrant angle.

The teacher may wish to derive the formula for cos (A – B) by using the Law of Cosines. Once this formula is established, the formulas for sines and cosines of the sum or difference of two angles, double angles, or half angles can be derived. Students should understand the interrelationship of the trigonometic functions and be able to make use of the formulas in various applications. The solution of equations such as $\cos 2x + \cos x + 1 = 0$ should be included.

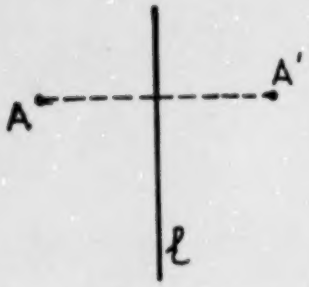
Teachers should note that a list of formulas appears in the *Course III* Regents Examination. The list is given in Appendix A of this publication. It is suggested that teachers make such a list available to their students throughout the year.

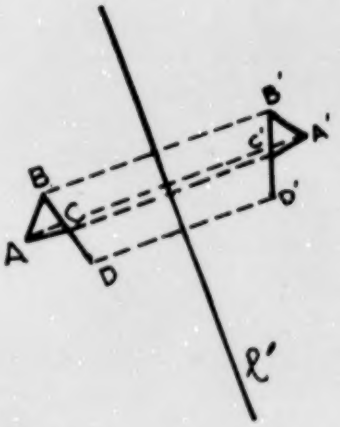
IV. Transformation Geometry

Many rewarding presentations are possible in the area of transformation geometry. This is a topic which can bring together many different strands of the discipline of mathematics. Depending on the abilities and interests of their students, some teachers may wish to stress a manipulative or discovery approach, while others will choose a more deductive presentation. Different emphases are possible with regard to the group-theoretical ideas inherent in the topic, as well as with regard to the ideas of similarity and congruence.

A. A transformation of the plane is a one-to-one mapping of the plane onto itself. The first example of a transformation in this section is the line reflection.

B. A reflection in line ℓ (suggested notation: r_ℓ) is a mapping which takes every point A of the plane into a point A' such that $\overline{AA'}$ is perpendicular to ℓ and the distance from A to ℓ equals the distance from ℓ to A'.



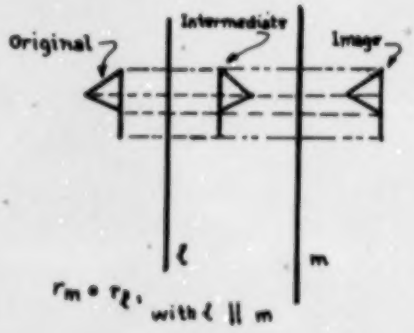


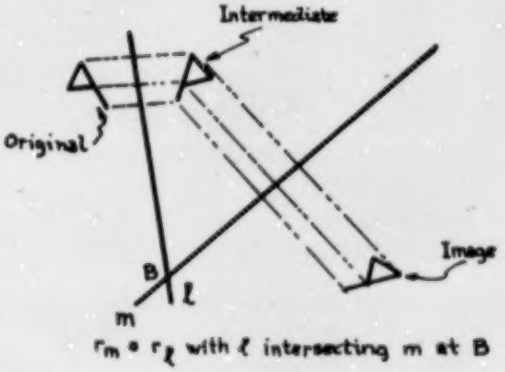
Flag figure A'B'C'D' is the image of ABCD under $r_{\ell'}$, reflection in line ℓ'.

Note that all points on line ℓ are mapped into themselves.

The line reflection preserves distance (i.e., AB = A'B') as well as betweenness (C between D and B yields C' between D' and B'), angle measure, and parallelism. The line reflection does not, however, preserve order (orientation); e.g., A, B, C are oriented clockwise, whereas A', B', C' are oriented counterclockwise.

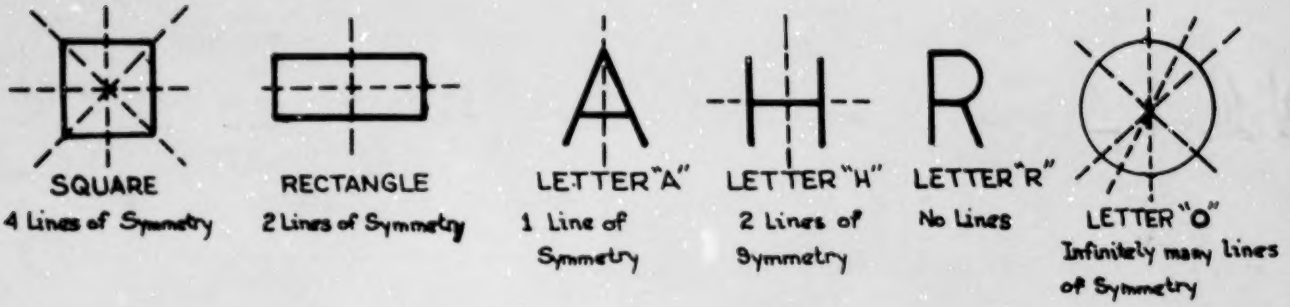
It is possible to consider the composition of two or more line reflections. Below are two illustrations. The notation $r_m \circ r_l$ denotes reflection in line l followed by reflection in line m.





Note that the square of a line reflection (i.e., the composition of a line reflection with itself) takes every point of the plane back into itself and is, therefore, an identity mapping.

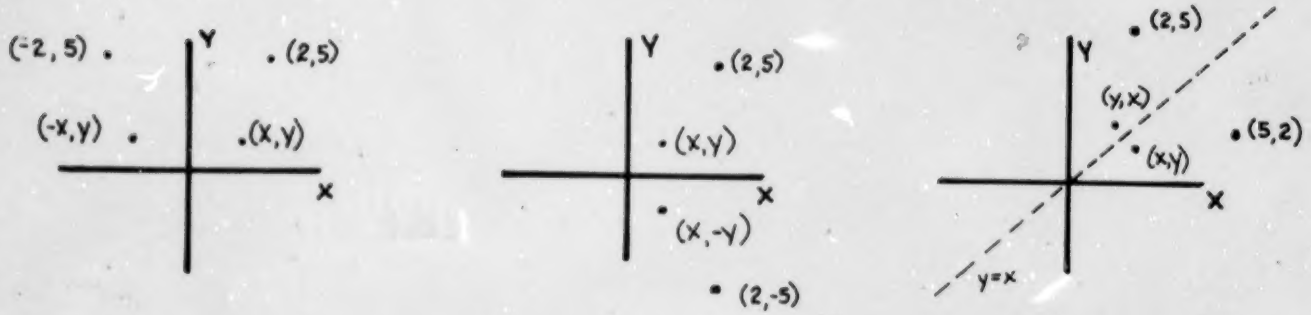
A figure which is its own image under some line reflection is said to have line-symmetry, or to be symmetric in, or about, the line. A figure may have more than one such symmetry.



A transformation which preserves distance is called an isometry. An isometry which preserves order is called a direct isometry. Thus, the line reflection is an example of an isometry, but not a direct isometry.

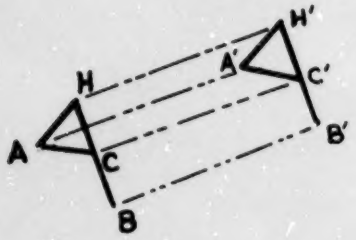
Consider the special case of the coordinatized x,y plane. Under a reflection in the y-axis, the image of point P with coordinates (x, y) is point P' with coordinates (− x, y). Or, using function notation: $r_{y\text{-axis}}$: (x, y) ⟶ (− x, y). Similarly,

$r_{x\text{-axis}}$: (x, y) ⟶ (x, − y) and $r_{x = y}$: (x, y) ⟶ (y, x)



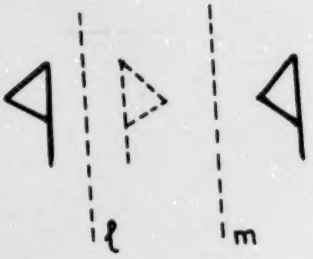
Students should be encouraged to discover these ideas through their own investigation. A good challenge for the more interested students is to try to find a similar rule for a reflection, say, in the line x = 2.

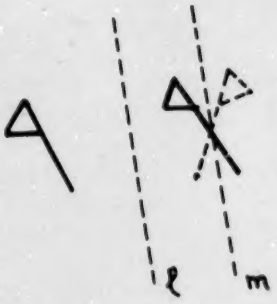
C. A translation is a mapping which assigns to every point in the plane another point in the plane such that every image point is located the same distance away, and in the same direction, from its pre-image, as every other. Or, more simply, a translation "moves" every point in the plane the same distance in the same direction.



All the points in flag-figure AHCB are "moved" (with every other point in the plane) equal distances in the same direction, in this case to A'H'C'B'.

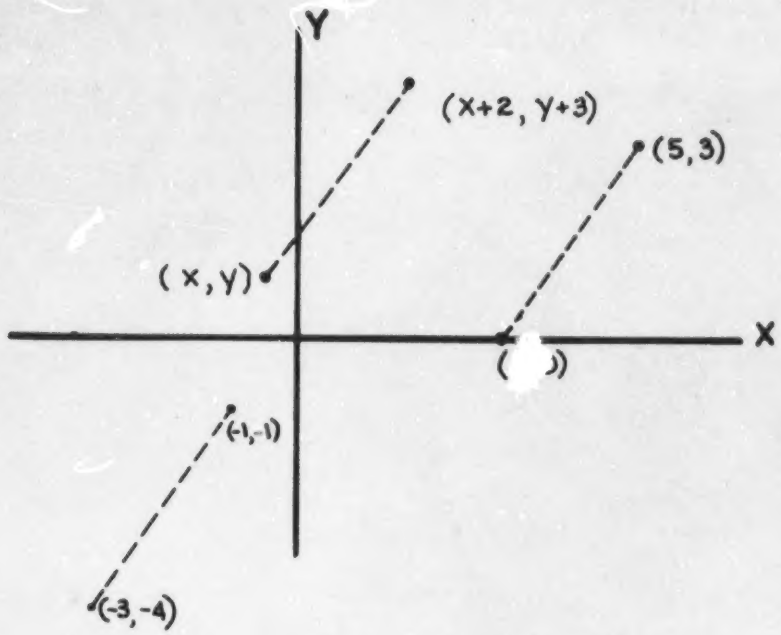
A translation can also be considered as the (non-unique) product of two line reflections in parallel lines (axes), as these diagrams illustrate:





Translation can be defined in this manner, although this approach is not suggested at this level. Note that since they preserve distance and order, translations are direct isometries.

With respect to a coordinatized x,y-plane, the translation which "moves" a point (x, y) a units in the x-direction, and b units in the y-direction (suggested notation: $T_{a,\ b}$) has the rule, $T_{a,\ b}$: $(x, y) \longrightarrow (x + a, y + b)$.

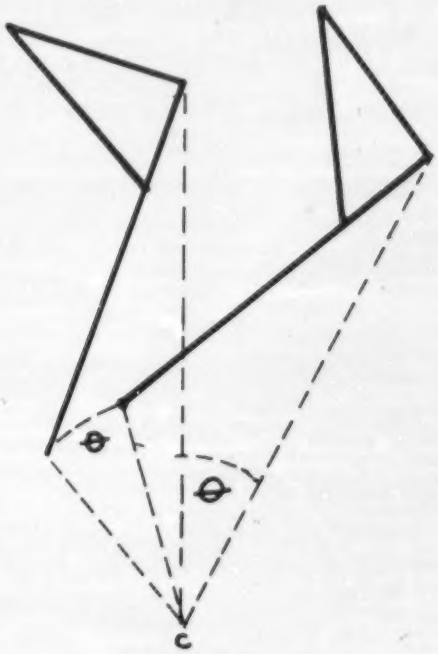


Images of points under translation $T_{2,\ 3}$

A translation which keeps every point where it is (i.e., maps every point onto itself; or, moves every point zero units in any direction) is the identity transformation. It can be called $T_{0,\ 0}$. Furthermore, for every translation $T_{a,\ b}$ there exists another translation $T_{-a,\ -b}$ which will take every image-point under $T_{a,\ b}$ back to where it started. Thus the products, or composition, $T_{-a,\ -b} \circ T_{a,\ b} = T_{0,\ 0}$; i.e., $T_{a,\ b}$ and $T_{-a,\ -b}$ are inverses of each other.

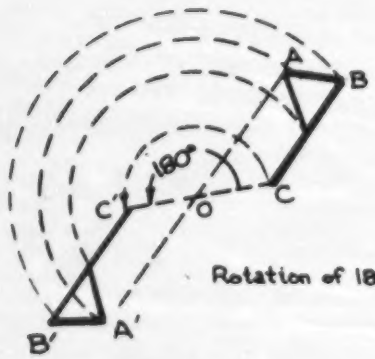
The idea of translationai symmetry is probably more difficult to grasp than line or rotational symmetry. For this reason, many teachers may wish not to introduce it except to their more talented students. A figure has translational symmetry if there exists a translation that maps the figure onto itself. Note that no figure of finite proportions can have translational symmetry. An example of translational symmetry is a wallpaper or other "never ending" pattern. Other examples are the graphs of the trigonometric functions.

D. A rotation R about center, c, and through an angle of θ (notation: $R_{c,\theta}$ or $Rot_{c,\theta}$) is a mapping which "moves" all points in the plane through an angle of θ with point c fixed at the vertex of the angle. (If θ is positive, the motion is counterclockwise. If θ is negative, the motion is clockwise.)

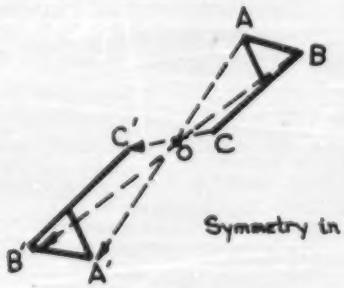


The center of the rotation is mapped to itself.

A rotation of 180 degress about a point P is sometimes called a <u>half-turn</u> or a <u>reflection in point P</u> (point-reflection).

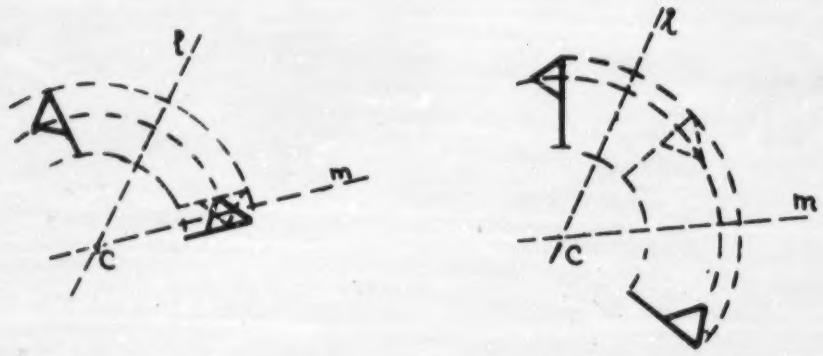


Rotation of 180° Same as

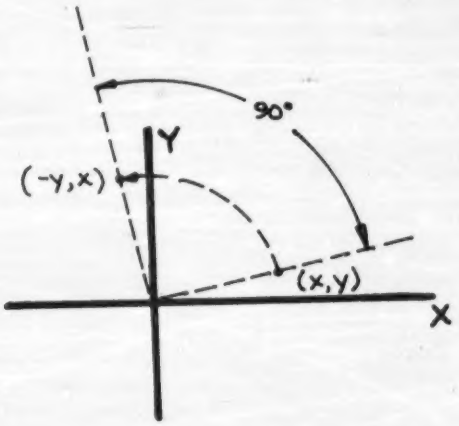


Symmetry in Point "O"

Note that a rotation about point C can be considered to be (and is often defined to be) the product of two line reflections whose axes intersect at C, as these diagrams illustrate:



Since rotations preserve distance and order, they are direct isometries. Discovery of rules for certain special rotations about the origin (90°, 180°, 270°, or -90°) is within the range of most students' abilities. For example, if $\theta = 90$, R_{90}: $(x, y) \rightarrow (-y, x)$ as this illustration shows. Note: If the center of rotation is the origin of the coordinate plane, then this is understood by writing simply R_{θ}.



A figure has rotational symmetry if there exists a rotation (other than the identity) which maps it onto itself. If the figure has the 180°-rotational symmetry, it may be said to have point-symmetry. (Most crossword puzzles, for example, are point symmetric.)

SQUARE
has point symmetry and other rotational symmetries

RECTANGLE
Point Symmetry

R

LETTER "R"
No rotational symmetry

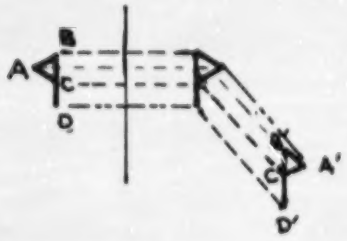
H

LETTER "H"
Point Symmetry

WINGED TRIAD
rotational (120° and 240°) symmetries

The set of rotations of a plane about a fixed point forms a group under composition. This group has several interesting subgroups; for example, the set of rotations through 30k degrees, where k is an integer.

E. Thus far, three types of transformation have been discussed: reflections, translations, and rotations, all of which are isometries. It can be shown that every isometry is either one of these, or a glide-reflection. A glide-reflection is the product of a reflection and a translation.

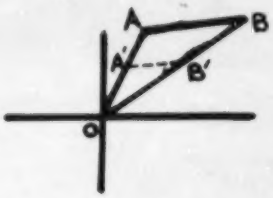


Flag figure ABCD mapped to figure A'B'C'D' under glide-reflection shown is shown to the left.

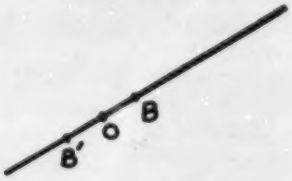
A glide-reflection can be seen to be the product of three line reflections (one for the reflection, two for the translation). Since translations and rotations are each equivalent to two line reflections, it can be seen that every isometry can be expressed as the composition of no more than three line reflections. This interesting result is often called the Three Line Reflection Theorem. Some teachers may wish to have their students "test it out" by "finding the reflections."

Probably most teachers would want to present an example of a transformation which is not an isometry. Perhaps the best example of such a mapping is the dilation. A dilation with center at the origin of a coordinate plane and scale factor a (notation: D_a) assigns to every point A another point A' on ray $\overrightarrow{OA}$ such that OA' = a(OA). Point 0 is mapped onto itself.

A mapped to A' under dilation D_2



$\Delta OAB \rightarrow \Delta OA'B'$ under $D_{\frac{1}{2}}$



$B \rightarrow B'$ under D_{-1}

Since distances are not preserved (for example, $OA \neq OA'$), dilations are not isometries. One exception is D_1 which maps the plane onto itself and is therefore the identity; the other is D_{-1} which is equivalent to a point reflection since it also preserves distance.

F. A study of transformation geometry can provide many applications to group theory. For example, since:

the composition of any two translations is a translation

it can be demonstrated that the composition of translations is associative

there exists an identity translation ($T_{0,0}$)

every translation has an inverse

The set of all translations, together with the operation composition, constitutes a group. That the group is also commutative is a useful investigation.

The set of rotations of a plane about a fixed point also forms a group under composition. This group has several interesting subgroups; for example, the set of rotations through 30k degrees, where k is an integer.

It can also be shown that the set of all isometries forms a group under composition and that the set of direct isometries forms a subgroup of this group.

These ideas may be pursued as an optional topic with more talented students.

The study of transformation geometry provides alternate definitions of congruence and similarity. Two figures are congruent if there exists an isometry under which one is the image of the other. Two figures are similar if there exists an isometry and a dilation under the composition of which one figure is mapped to the other. Teachers are encouraged to discuss fully this notion of congruence and to compare and contrast it with the concepts students have already met in their study of Euclidean geometry. Some teachers may want to have their classes develop isometry-oriented proofs for theorems, or other propositions, previously justified in other ways. Hopefully, students will come to agree that looking at a mathematical question from more than one point of view, or with more than one set of assumptions and definitions, is a valuable and enjoyable experience for mathematics students.

V. Probability and Statistics

A. B. In previous courses, students have studied simple probability problems such as finding the probability of a "6" on a single toss of a fair die. Such problems may be reviewed along with other aspects of this material from *Course I* and *Course II* such as the counting principle and elementary combinatorics. This study may now be extended to situations involving the probability of exactly, at least, or at most r successes in n trials.

Example 1: Find the probability of rolling one "6" in six tosses of a fair die.

If the problem were, "Find the probability of the event E of rolling a fair die six times and obtaining 6 on the first roll and some other number on each of the others," then

$P(E) = \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5$ since the probability of a 6 is $\frac{1}{6}$ and the probability of some other number is $\frac{5}{6}$. But in the original problem, the 6 can be either on the first toss, the second, and so on, up to the sixth. Since the one required 6 can occur in any one of six places, and since the probability of its occurrence in any one of these places is the same, it can be concluded that:

$P(\text{exactly one 6 in six tosses}) = 6 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 = \frac{3125}{7776} \approx .402$

Students may compare this with empirical results.

To find the probability of obtaining exactly two 6's in six tosses, consider initially the probability that the first two tosses, only, will be 6's. This is given by:

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 \approx .0134$$

But the 6's in the problem do not have to be the first two. Two things (for example, two 6's) can be distributed among six places in ${}_6C_2$ ways. Therefore, the result is:

P(exactly two 6's in six tosses) =

$${}_6C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 = 15 \cdot \frac{1}{36} \cdot \frac{625}{1296} \approx .201$$

It follows that the probability of obtaining exactly r 6's $(0 < r < 6)$ in six tosses is ${}_6C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{6-r}$ and of obtaining r 6's in n tosses is ${}_nC_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{n-r}$.

Finally, it follows that in the case of an experiment where there are exactly two outcomes, (6 or not 6; heads or tails; American League or National League; etc.), with probability of "success" p and probability of "failure" $q = 1 - p$, that the probability of r successes in n trials is $\binom{n}{r} p^r q^{n-r}$.

Experiments with exactly two outcomes are called Bernoulli experiments (after James Bernoulli, 1654-1705). The individual "tosses" discussed here are examples of independent trials of a Bernoulli experiment.

Example 2: The probability, p, of obtaining at least eight heads in ten tosses of a fair coin is equal to the sum of the probabilities of obtaining exactly eight, exactly nine, and exactly ten heads.

$$P = \binom{10}{8}\left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + \binom{10}{9}\left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + \binom{10}{10}\left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

Example 3: If the coin is "loaded" so that P (heads) $\frac{1}{3}$, P (tails) $= \frac{2}{3}$, then the probability of r heads in n tosses is $\binom{n}{r}\left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{n-r}$

C. With this background, the student is prepared to move on to a study of the Binomial Theorem. There are a variety of approaches to this theorem, but it is hoped that whatever approach is used, students will come to see the theorem as an integral part of significant mathematical situations rather than as an end in itself. It is also hoped that whatever notation will be involved in expressing the theorem will have become so reasonable and ordinary to most students, that it will not act as a barrier to understanding.

Students may be encouraged to look for patterns among coefficients of $(a + b)^n$ for n = 1, 2, 3, ... and may discover or be led to observe that these coefficients are the entries in Pascal's Triangle as well as different values of ${}_nC_r$ for various n and r.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

etc.

Students might now be asked to reconsider the problem of tossing a coin - not necessarily a fair one - with probability of heads p and probability of tails q. If the coin is tossed n times, a chart, similar to the one on the following page, might be developed for different values of n.

n	Possible Outcomes	Probability of Outcomes	Probability of r heads $0 \leq r \leq n$	
1	(H)	p	$r = 1$ $P(1 \text{ head}) = p$	$p + q = 1$
	(T)	q	$r = 0$ $P(0 \text{ heads}) = q$	
2	(H,H)	$pp = p^2$	$r = 2$ $P(2 \text{ heads}) = p^2$	$p^2 + 2pq + q^2 = 1$
	(H,T)	pq	$r = 1$ $P(1 \text{ head}) = 2pq$	
	(T,H)	$qp = pq$		
	(T,T)	$qq = q^2$	$r = 0$ $P(0 \text{ heads}) = q^2$	
3	(H,H,H)	p^3	$r = 3$ $P(3 \text{ heads}) = p^3$	$p^3 + 3p^2q + 3pq^2 + q^3 = 1$
	(H,H,T)	$ppq = p^2q$		
	(H,T,H)	$pqp = p^2q$	$r = 2$ $P(2 \text{ heads}) = 3p^2q$	
	(T,H,H)	$qpp = p^2q$		
	(T,T,H)	$qqp = pq^2$		
	(T,H,T)	$qpq = pq^2$	$r = 1$ $P(1 \text{ head}) = 3pq^2$	
	(H,T,T)	$pqq = pq^2$		
	(T,T,T)	q^3	$r = 0$ $P(0 \text{ heads}) = q^3$	
4	(H,H,H,H)	p^4	$r = 4$ $P(4 \text{ heads}) = p^4$	$p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4 = 1$
	↓	↓	$r = 3$ $P(3 \text{ heads}) = 4p^3q$	
	↓	↓	$r = 2$ $P(2 \text{ heads}) = 6p^2q^2$	
	↓	↓	$r = 1$ $P(1 \text{ head}) = 4pq^3$	
	(T,T,T,T)	q^4	$r = 0$ $P(0 \text{ heads}) = q^4$	
5	ETC.			

The entries in the right column of the table suggest a statement of the Binomial Theorem:

$$(p + q)^n = \binom{n}{0} p^n + \binom{n}{1} p^{n-1}q + \binom{n}{2} p^{n-2}q^2 + \ldots + \binom{n}{n} q^n$$

Since p and q can represent any algebraic expressions, the formula can be used to do things like finding the expansion of $(\frac{1}{2}x - 2y)^5$.

Depending on the skills and interests of their classes, teachers may wish to continue with this topic through a more detailed discussion of the binomial distribution. For more capable students, transition into the realm of statistics could be, for example, a demonstration that for a binomial distribution (n trials, probability of success p) the mean is equal to the product np.

D. E. In *Course I*, students have discussed three statistical measures: the mean, median and mode. (See outline and notes for *Course I*.) While these measures, especially the mean, are very useful measures of central tendency, it is clear that for certain kinds of investigation, they are insufficient. For example, suppose that one thousand high school students are given some standardized test as part of a special research project, and suppose that the mean of the scores is 80. This might seem to be directly usable information in and of itself until it is realized that this mean of 80 could have been the result either of all students scoring approximately 80 on the test or as a result of half scoring 60 and half scoring 100. Obviously, this is quite different in terms of the research project. Teachers can easily make up many examples where the measures of central tendency already studied are insufficient sources of usable information about a set of numbers. Therefore, in addition to a measure of central tendency, some measure of the degree of deviation from that central tendency, or of dispersion, is necessary.

One of the simplest measures of dispersion is the range, which is the difference between the highest and lowest numbers in the set. Sometimes this measure is useful but since it depends on only two numbers, it is easy to see circumstances in which it would be inadequate. Some students might suggest that a measure of dispersion using all the numbers might be the sum of the deviations, $(\bar{x} - x_i)$, of each measure from the mean, $\bar{x}$, but of course, this sum turns out to be zero. However, if the sign of each difference is disregarded, then each of the differences would be nonnegative and their sum different from zero (except in the rare case when all the data are equal). The measure, in symbols, $\frac{1}{n}\sum_{i=1}^{n} |\bar{x} - x_i|$, is called the mean

absolute deviation. ($\sum$ - notation will be discussed later.) A disadvantage of this measure is its unwieldiness when large numbers of data are being considered.

A more widely used measure of dispersion is the variance:

$$\text{variance of a set of n numbers} = \frac{1}{n}\sum_{i=1}^{n}(\bar{x} - x_i)^2 .$$

Here each difference of a number from the mean is squared, thus assuring that the sum will be nonnegative. The sum of these squared deviations is divided by n, the total number of measures, to keep this measure at a manageable size.

An objection to the variance as a measure of dispersion is the fact that it does include an element of distortion stemming from the fact that before the division, each difference was squared. One further step leads to what is probably the most important and widely used measure of dispersion, the standard deviation, the square root of the variance:

$$\text{s.d.} = \frac{1}{n}\sum_{i-1}^{n}(\bar{x} - x_i)^2$$

where n is the total number of measures and $\bar{x}$ is the mean. (Other commonly used symbols for standard deviation are σ, and s.) The variance of a set of measures is often symbolized by s^2 to correspond to the use of s for standard deviation. Students should be asked to work out several problems in computing standard deviation, some involving fairly large numbers of measurements.

F. Since it is very difficult to deal with certain statistical investigations, consisting as they generally do of large accumulations of data, without a system of notation to make things more manageable and efficient, $\sum$ - notation has been introduced in this outline. Teachers may find the discussion of $\sum$ - notation a useful introduction to this material on statistics.

The $\sum$ (Greek capital sigma) means, in these applications, "take the sum of all of the following" and is often used with some indexing letters i, j, etc. This discussion will be limited to one index at a time, usually i.

Example 1: $\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 12$

Example 2: A certain survey has produced these fifteen measures: 7, 1, 7, 3, 10, 7, 0, 1, 3, 6, 4, 0, 7, 7, 3.

If we index them in this way:

$7_1, 1_2, 7_3, 3_4, 10_5, 7_6, 0_7, 1_8, 3_9, 6_{10}, 4_{11}, 0_{12}, 7_{13}, 7_{14}, 3_{15}$,

then $\sum_{i=1}^{15} x_i$ = the sum of these measures in the order given = 66. Often it is wise to order the measures first before indexing and taking their sum:

$0_1, 0_2, 1_3, 1_4, 3_5, 3_6, 3_7, 4_8, 6_9, 7_{10}, 7_{11}, 7_{12}, 7_{13}, 7_{14}, 10_{15}$.

In some cases it might prove efficient to group the measures according to their frequency of occurrence. The table below does this for the set of measures given above.

index	measure	frequency
1	0	2
2	1	2
3	3	3
4	4	1
5	6	1
6	7	5
7	10	1

From this table, we see that the first measure, 0, occurs with a corresponding first frequency 2; that the second measure, 1, occurs with a corresponding second frequency 2; and so on. Then, if we use "f" to represent "frequency" we may index the seven different measures and their corresponding frequencies: for $i = 1$, $f_i = 2$ and $x_i = 0$; $i = 2$, $f_i = 2$ and $x_i = 1$; ...; for $i = 7$, $f_i = 1$ and $x_i = 10$. Then

$$\sum_{i=1}^{7} f_i x_i = 2(0) + 2(1) + 3(3) + 1(4) + 1(6) + 5(7) + 1(10) = 66.$$

Such an organization is clearly quite useful where large numbers of repeated data are being computed.

If the symbol $\bar{x}$ (read "x-bar") is used to represent the mean, $\sum_{i=1}^{n} (\bar{x} - x_i)$ represents the sum of the differences of n measure from their mean, while the summation $\sum_{i=1}^{n} (\bar{x} - x_i)^2$ is the sum of the squares of the difference of the measures from the mean. Of course, this could also be written

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 .$$

Sometimes, when working with a set of numbers whose limits everyone knows quite well, the limits of summation are omitted. Thus, in this case, the sum given immediately above might be written $\sum (x_i - \bar{x})^2$. Since notation is shorthand for an ordinary sum, the familiar properties of addition and multiplication — associativity, commutativity and distributivity — apply. Thus, if c is a constant, $\sum cx_i = c\sum x_i$ and, for example,

$$\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i .$$

These statements can be readily demonstrated by writing out the sum, rearranging the terms, and resumming.

Example: Calculate the mean absolute deviation, variance (s^2), and standard deviation (s) of the following set of data:

9, 16, 9.5, 12, 12, 9, 16, 16, 17.5, 19, 16, 12, 13, 12, 13, 14, 13, 16, 19, 16

Solution: Of the n measures, there are k different ones (i.e., the n measures can be grouped into k classes) each occurring with its own frequency. In this case, n = 20, k = 8. The mean is 14.

i Index	x_i Measure	f_i Frequency	$\lvert\bar{x} - x_i\rvert$	$f_i\lvert\bar{x} - x_i\rvert$	$(\bar{x} - x_i)^2$	$f_i\ (\bar{x} - x_i)^2$
1	9	2	5	10	25	50
2	9.5	1	4.5	4.5	20.25	20.25
3	12	4	2	8	4	16
4	13	3	1	3	1	3
5	14	1	0	0	0	0
6	16	6	2	12	4	24
7	17.5	1	3.5	3.5	12.25	12.25
8	19	2	5	10	25	50
				$\sum_{i=1}^{8} f_i\lvert\bar{x}-x_i\rvert$		$\sum_{i=1}^{8} f_i(\bar{x}-x_i)^2$
				51		175.5

Mean absolute deviation = $\frac{51}{20} = 2.55$

Variance = $\frac{175.5}{20} = 8.775$

Standard deviation $\approx$ 2.96.

Of course, students will probably be asked to deal with problems involving far more than twenty measurements.

The computation of variance and standard deviation as defined requires computation of $\bar{x} - x_i$ for all measures and squaring those differences. It can be shown, however, that the following equality holds:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$ and more skillful or well-motivated groups may wish to consider this demonstration:

$$\frac{1}{n}\sum (x_i - \bar{x})^2 = \frac{1}{n}\sum [x_i^2 - 2x_i\bar{x} + \bar{x}^2]$$

$$= \frac{1}{n}\left[\sum x_i^2 - 2\bar{x}\sum x_i + \sum \bar{x}^2\right]$$

$$= \frac{1}{n}\sum x_i^2 - \left(\frac{1}{n} \cdot 2 \cdot \bar{x} \cdot n\bar{x}\right) + \frac{1}{n} \cdot n\bar{x}^2$$

$$= \frac{1}{n}\sum x_i^2 - 2\bar{x}^2 + \bar{x}^2$$

$$= \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

This formulation is especially useful if the measures are such that a limited number of them occur with some frequency. For, using the fact that for n measures grouped into k classes, $\sum_{i=1}^{k} f_i x_i = \sum_{i=1}^{n} x_i$, it can be shown that $s^2 = \frac{1}{n}\sum_{i=1}^{k} f_i \cdot (x_i)^2 - \bar{x}^2$.

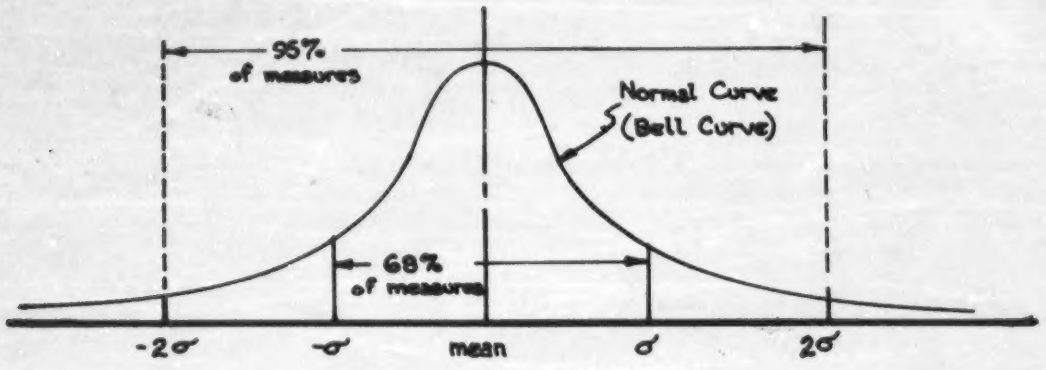
Example: For the set of measures given above, s^2 can be computed as

$$s^2 = \frac{1}{20}\sum_{i=1}^{8} f_i\,(x_i)^2 - 14^2$$

$$= \frac{4095.50}{20} - 196 = 8.775$$

Another useful problem for especially able students would be to show that for a binomial distribution, $s = \sqrt{npq}$.

All classes might find it useful to be introduced to some facts about "bell-curves," although few of them can be proved at this point. In a set of measures whose distribution is described by a bell curve — which formally is a normal distribution — 68 percent of the measures fall within one standard deviation of the mean and 95 percent within two standard deviations.



Teachers may find a discussion of such distributions, applied for example to examinations students have taken, a good illustration of statistical ideas.

VI. Optional Topics

As time permits, teachers may wish to explore the following topics:

A. Polar coordinates
B. Number theory
C. Mathematical induction
D. Sequences and series
E. Matrices
F. Synthetic substitution, remainder theorem, and factor theorem

APPENDIX A: FORMULAS

Pythagorean and Quotient Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\tan^2 A + 1 = \sec^2 A$$
$$\cot^2 A + 1 = \csc^2 A$$

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

Functions of the Sum of Two Angles

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Functions of the Difference of Two Angles

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Functions of the Double Angle

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos 2A = 2 \cos^2 A - 1$$
$$\cos 2A = 1 - 2 \sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Functions of the Half Angle

$$\sin \tfrac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{2}}$$

$$\cos \tfrac{1}{2}A = \pm \sqrt{\frac{1 + \cos A}{2}}$$

$$\tan \tfrac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Area of Triangle

$$K = \tfrac{1}{2} ab \sin C$$

Standard Deviation

$$s = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\bar{x} - x_i)^2}$$

APPENDIX B: NOTATIONS

Relations and Functions: F^{-1} the function which is the inverse of the function F

Inverse Trig. Functions: Arc sin, Arc cos, etc.

Composition of Mapping: $f \circ g$ means g followed by f — "do g first"

Transformations: r_ℓ reflection in line ℓ

$R_{c,90}$ or $Rot_{c,90}$ rotation counter-clockwise about center c

$R_{c,-90}$ or $Rot_{c,-90}$ rotation clockwise about center c

R_{90} or Rot_{90} rotation about <u>origin</u> of 90°

$T_{a,b}$ translation a units horizontally, b units vertically

D_a dilation with center at origin and scale factor a of a coordinate plane

Prob./Stat./Combinatorics: Three notations, ${}_nC_r$, C(n,r) and $\binom{n}{r}$ appear in *Course II*

In *Course III*, teachers may use any of these, but only ${}_nC_r$ and $\binom{n}{r}$ appear in the outline.

$\bar{x}$ mean

$\bar{x} - x_i$ difference of a measure x_i from mean $\bar{x}$

s^2 variance

s standard deviation, but s.d., σ are also used

$\sum_{i=1}^{n}$ the sum of n numbers beginning with the first and ending with the nth.